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STARLIKENESS PROBLEMS FOR CERTAIN ANALYTIC FUNCTIONS CONCERNED WITH SUBORDINATIONS

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ABSTRACT. Let \mathcal{A}_n be the class of functions $f(z)$ which are analytic in the open unit disk \mathbb{U} with $f(0) = 0$, $f'(0) = 1$, $f''(0) = f'''(0) = \dots = f^{(n)}(0) = 0$ and $f^{(n+1)}(0) \neq 0$. Applying the results due to S. S. Miller (J. Math. Anal. Appl. **65**(1978), 289-305), some interesting starlikeness problems concerned with subordinations are discussed. The results in the paper are extensions of results by M. Obradović (Hokkaido Math. J. **27**(1998), 329-335).

1. INTRODUCTION

Let $\mathcal{H}[a_0, n]$ denote the class of functions $p(z)$ of the form

$$p(z) = a_0 + \sum_{k=n}^{\infty} a_k z^k \quad (n = 1, 2, 3, \dots)$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, where $a_0 \in \mathbb{C}$.

Also, let \mathcal{A}_n denote the class of functions

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots \quad (n = 1, 2, 3, \dots)$$

that are analytic in \mathbb{U} with $a_{n+1} \neq 0$ and $\mathcal{A} \equiv \mathcal{A}_1$.

If $f(z) \in \mathcal{A}_n$ satisfies

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U})$$

for some real α ($0 \leq \alpha < 1$), then we say that $f(z)$ is starlike of order α and written by $f(z) \in \mathcal{S}^*(\alpha)$ and $\mathcal{S}^* \equiv \mathcal{S}^*(0)$.

Let $f(z)$ and $g(z)$ be analytic in \mathbb{U} . Then $f(z)$ is said to be subordinate to $g(z)$ if there exists an analytic function $w(z)$ in \mathbb{U} satisfying $w(0) = 0$, $|w(z)| < 1$ ($z \in \mathbb{U}$) and such that $f(z) = g(w(z))$. We denote this subordination by

$$f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

In particular, if $g(z)$ is univalent in \mathbb{U} , then the subordination

$$f(z) \prec g(z) \quad (z \in \mathbb{U})$$

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is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ (cf. [3]).

The basic tool in proving our results is the following lemma due to Miller and Mocanu [2] (also [3]).

Lemma 1. *Let the function $w(z)$ defined by*

$$w(z) = a_n z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots \quad (n = 1, 2, 3, \dots)$$

be analytic in \mathbb{U} with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathbb{U}$, then there exists a real number $k \geq n$ such that

$$\frac{z_0 w'(z_0)}{w(z_0)} = k.$$

2. MAIN RESULT

Applying Lemma 1, we have the following lemma.

Lemma 2. *Let $p(z) \in \mathcal{H}[1, n]$ satisfy the condition*

$$p(z) - \frac{1}{\mu} z p'(z) \prec 1 + \lambda z \quad (z \in \mathbb{U})$$

for some complex number μ ($\operatorname{Re}(\mu) < n, \mu \neq 0$) and some complex number λ ($0 < |\lambda| \leq 1$), then

$$p(z) \prec 1 + \lambda_1 z \quad (z \in \mathbb{U}),$$

where λ_1 is a complex number such that

$$|\lambda_1| = |\lambda| \frac{|\mu|}{|n - \mu|}. \quad (2.1)$$

Proof. We consider the function $p(z)$ defined by

$$p(z) = 1 + \lambda_1 w(z)$$

with λ_1 is given by (2.1). Then, $w(z)$ is analytic in \mathbb{U} and $w(0) = 0$. We want to show that $|w(z)| < 1$ ($z \in \mathbb{U}$). If there exists a point $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$, then we can write $z_0 w'(z_0) = k w(z_0)$ ($k \geq n$) by Lemma 1. Putting $w(z_0) = e^{i\theta}$,

we get

$$\begin{aligned}
\left| p(z_0) - \frac{1}{\mu} z_0 p'(z_0) - 1 \right| &= \left| \lambda_1 w(z_0) - \frac{1}{\mu} \lambda_1 z_0 w'(z_0) \right| \\
&= \left| \lambda_1 e^{i\theta} - \frac{1}{\mu} \lambda_1 k e^{i\theta} \right| \\
&= |\lambda_1| \frac{|\mu - k|}{|\mu|} \\
&= |\lambda_1| \frac{\sqrt{(k - \operatorname{Re}(\mu))^2 + (\operatorname{Im}(\mu))^2}}{|\mu|} \\
&\geq |\lambda_1| \frac{\sqrt{(n - \operatorname{Re}(\mu))^2 + (\operatorname{Im}(\mu))^2}}{|\mu|} \\
&= |\lambda_1| \frac{|n - \mu|}{|\mu|} = |\lambda|.
\end{aligned}$$

This contradicts the assumption of Lemma 2. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This implies that $|w(z)| < 1$ ($z \in \mathbb{U}$). This completes the proof of the lemma. \square

Next, we show

Lemma 3. *Let λ and λ_1 be complex numbers such that $0 < |\lambda_1| < |\lambda| < 1$ and let $Q(z) \in \mathcal{H}[1, n]$ such that*

$$Q(z) \prec 1 + \lambda_1 z \quad (z \in \mathbb{U}). \quad (2.2)$$

(1) *If $p(z) \in \mathcal{H}[1, n]$ and*

$$Q(z)[\alpha + (1 - \alpha)p(z)] \prec 1 + \lambda z \quad (z \in \mathbb{U}) \quad (2.3)$$

for some real α such that

$$\alpha \leq \begin{cases} \frac{1 - |\lambda|}{1 + |\lambda_1|} & (0 < |\lambda| + |\lambda_1| \leq 1) \\ \frac{1 - (|\lambda|^2 + |\lambda_1|^2)}{2(1 - |\lambda_1|^2)} & (|\lambda|^2 + |\lambda_1|^2 \leq 1 \leq |\lambda| + |\lambda_1|), \end{cases} \quad (2.4)$$

then $\operatorname{Re}(p(z)) > 0$.

(2) *If $w(z) \in \mathcal{H}[0, n]$ and*

$$Q(z)[1 + w(z)] \prec 1 + \lambda z \quad (z \in \mathbb{U}), \quad (2.5)$$

then

$$|w(z)| < \frac{|\lambda| + |\lambda_1|}{1 - |\lambda_1|} \leq 1, \quad (2.6)$$

where $|\lambda| + 2|\lambda_1| \leq 1$.

The bound (2.6) and the value of α given by (2.4) are best possible.

Proof. To prove (1), we think about

$$g(z) = \frac{1 + \lambda z}{1 + \lambda_1 e^{i\theta} z} \quad (z \in \mathbb{U})$$

for some real θ .

It follows that

$$\left| g(z) - \frac{1 - \lambda \bar{\lambda}_1 r^2 e^{-i\theta}}{1 - |\lambda_1|^2 r^2} \right| \leq \frac{|\lambda - \lambda_1 e^{i\theta}| r}{1 - |\lambda_1|^2 r^2}$$

for $|z| \leq r < 1$.

If we put $\lambda = |\lambda| e^{i\theta_0}$, $\lambda_1 = |\lambda_1| e^{i\theta_1}$ and $\phi = \theta_0 - \theta_1 - \theta$, we can rewrite

$$\left| g(z) - \frac{1 - |\lambda| |\lambda_1| r^2 e^{i\phi}}{1 - |\lambda_1|^2 r^2} \right| \leq \frac{||\lambda_1| - |\lambda| e^{i\phi}| r}{1 - |\lambda_1|^2 r^2} \quad (|z| \leq r < 1).$$

Hence for all r ($0 < r < 1$), we obtain

$$\operatorname{Re}(g(z)) > \frac{1 - |\lambda| |\lambda_1| \cos \phi - \sqrt{|\lambda|^2 - 2|\lambda| |\lambda_1| \cos \phi + |\lambda_1|^2}}{1 - |\lambda_1|^2} \equiv h(\phi).$$

Note that

$$h'(\phi) = \frac{|\lambda| |\lambda_1| \sin \phi}{1 - |\lambda_1|^2} \left(1 - \frac{1}{\sqrt{|\lambda|^2 - 2|\lambda| |\lambda_1| \cos \phi + |\lambda_1|^2}} \right).$$

Therefore, $h'(\phi) = 0$ if $\sin(\phi) = 0$ or $\sqrt{|\lambda|^2 - 2|\lambda| |\lambda_1| \cos \phi + |\lambda_1|^2} = 1$. Since

$$|\lambda|^2 - 2|\lambda| |\lambda_1| \cos \phi + |\lambda_1|^2 \leq (|\lambda| + |\lambda_1|)^2,$$

if $0 < |\lambda| + |\lambda_1| \leq 1$, then $h'(\phi) = 0$ for $\phi = 2k\pi$ ($k = 0, \pm 1, \pm 2, \dots$) and $\phi = (2k+1)\pi$ ($k = 0, \pm 1, \pm 2, \dots$). With this conditions, we see that

$$h(\phi) \geq h(2k+1)\pi = \frac{1 - |\lambda|}{1 + |\lambda_1|} \geq \alpha.$$

If $\sqrt{|\lambda|^2 - 2|\lambda| |\lambda_1| \cos \phi + |\lambda_1|^2} = 1$, then $h'(\phi) = 0$ for $\phi = \phi_1$ such that

$$|\lambda|^2 - |\lambda| |\lambda_1| \cos \phi_1 + |\lambda_1|^2 = 1.$$

Thus, we have that

$$h(\phi) \geq h(\phi_1) = \frac{1 - (|\lambda|^2 + |\lambda_1|^2)}{2(1 - |\lambda_1|^2)} \geq \alpha$$

with $|\lambda|^2 + |\lambda_1|^2 \leq 1 \leq |\lambda| + |\lambda_1|$.

On the other hand, in view of (2.2) and (2.3), we have

$$\alpha + (1 - \alpha)p(z) \prec g(z) \quad (z \in \mathbb{U}).$$

So, we can lead $\operatorname{Re}(p(z)) > 0$.

To prove (2), note that in view of (2.2) and (2.5) we have

$$\begin{aligned} |w(z)| &= \left| \frac{(Q(z)(1+w(z)) - 1) - (Q(z) - 1)}{Q(z)} \right| \\ &\leq \frac{|Q(z)(1+w(z)) - 1| + |Q(z) - 1|}{|Q(z)|} \\ &\leq \frac{|\lambda| + |\lambda_1|}{|1 + \lambda_1 z|} \\ &< \frac{|\lambda| + |\lambda_1|}{1 - |\lambda_1|} \leq 1 \quad (z \in \mathbb{U}). \end{aligned}$$

Let us show the sharpness.

We take

$$Q(z) = 1 + \lambda_1 w_1(z) \quad (z \in \mathbb{U})$$

where $w_1(z)$ is analytic in \mathbb{U} such that $w_1(0) = 0$ and $|w_1(z)| < 1$ ($z \in \mathbb{U}$).

In the case of (1), we put

$$Q(z)[\alpha + (1 - \alpha)p(z)] = 1 + \lambda w_0(z) \quad (z \in \mathbb{U})$$

where $w_0(z)$ is analytic in \mathbb{U} such that $w_0(0) = 0$ and $|w_0(z)| < 1$ ($z \in \mathbb{U}$). Since

$$p(z) = \frac{1}{1 - \alpha} \left(\frac{1 + \lambda w_0(z)}{1 + \lambda_1 w_1(z)} - \alpha \right),$$

if we consider $w_0(z)$ such that $\lambda w_0(z) = -|\lambda|$ and $\lambda_1 w_1(z) = |\lambda_1|$, then we have

$$\alpha = \frac{1 - |\lambda|}{1 + |\lambda_1|} \quad (0 < |\lambda| + |\lambda_1| \leq 1).$$

Furthermore, if we take $w_0(z)$ such that $\lambda w_0(z) = |\lambda|e^{i\theta_0}$ and $w_1(z)$ such that $\lambda_1 w_1(z) = |\lambda_1|e^{i\theta_1}$, then we obtain that

$$\alpha = \frac{1 - (|\lambda|^2 + |\lambda_1|^2)}{2(1 - |\lambda_1|^2)}$$

with

$$|\lambda| \sin \theta_0 - |\lambda_1| \sin \theta_1 + |\lambda||\lambda_1| \sin(\theta_0 - \theta_1) = 0$$

and

$$\begin{aligned} &2(1 - |\lambda_1|^2)|\lambda| \cos \theta_0 + 2|\lambda|^2|\lambda_1| \cos \theta_1 + 2(1 - |\lambda_1|^2)|\lambda||\lambda_1| \cos(\theta_0 - \theta_1) \\ &= (|\lambda_1|^2 - 1)(1 + |\lambda|^2 - |\lambda_1|^2). \end{aligned}$$

In the case of (2), we put

$$Q(z)[1 + w(z)] = 1 + \lambda w_0(z) \quad (z \in \mathbb{U})$$

where $w_0(z)$ that $w_0(0) = 0$ and $|w_0(z)| < 1$ ($z \in \mathbb{U}$).

Then, we can obtain

$$w(z) = \frac{\lambda w_0(z) - \lambda_1 w_1(z)}{1 + \lambda w_0(z)} \quad (z \in \mathbb{U}).$$

If we take

$$w(z) = \frac{|\lambda| + |\lambda_1|}{1 - |\lambda_1|} \quad (z \in \mathbb{U}),$$

then we have $\lambda w_0(z) = |\lambda|$ and $\lambda_1 w_1(z) = -|\lambda_1|$. □

If we consider some real λ , λ_1 and fixed α in Lemma 3, we obtain Corollary 1 due to S. Ponnusamy and V. Singh [5].

Corollary 1. *Let λ and λ_1 be real with $0 < \lambda_1 < \lambda < 1$ and let $Q(z) \in \mathcal{H}[1, n]$ satisfy*

$$(1) \quad \begin{aligned} & Q(z) \prec 1 + \lambda_1 z \quad (z \in \mathbb{U}). \\ & \text{If } p(z) \in \mathcal{H}[1, n] \text{ and} \\ & Q(z)[\alpha + (1 - \alpha)p(z)] \prec 1 + \lambda z \quad (z \in \mathbb{U}) \end{aligned}$$

for some real α such that

$$\alpha = \begin{cases} \frac{1 - \lambda}{1 + \lambda_1} & (0 < \lambda + \lambda_1 \leq 1) \\ \frac{1 - (\lambda^2 + \lambda_1^2)}{2(1 - \lambda_1^2)} & (\lambda^2 + \lambda_1^2 \leq 1 \leq \lambda + \lambda_1), \end{cases} \quad (2.7)$$

then $\operatorname{Re}(p(z)) > 0$.

$$(2) \quad \begin{aligned} & \text{If } w(z) \in \mathcal{H}[0, n] \text{ and} \\ & Q(z)[1 + w(z)] \prec 1 + \lambda z \quad (z \in \mathbb{U}), \end{aligned}$$

then

$$|w(z)| < \frac{\lambda + \lambda_1}{1 - \lambda_1} \leq 1, \quad (2.8)$$

where $\lambda + 2\lambda_1 \leq 1$.

The bound (2.8) and the value of α given by (2.7) are best possible.

By virtue of Lemma 2, we deduce the sufficient condition for the class \mathcal{S}^* .

Theorem 1. *If $f(z) \in \mathcal{A}_n$ satisfies the condition*

$$f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} \prec 1 + \lambda z \quad (z \in \mathbb{U})$$

for some complex numbers μ ($\operatorname{Re}(\mu) < n$) and λ such that

$$0 < |\lambda| \leq \frac{|n - \mu|}{\sqrt{|n - \mu|^2 + |\mu|^2}}, \text{ then } f(z) \in \mathcal{S}^*.$$

Proof. If we put

$$Q(z) = \left(\frac{z}{f(z)} \right)^\mu,$$

then we get

$$Q(z) - \frac{1}{\mu} Q'(z) = f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} \prec 1 + \lambda z \quad (z \in \mathbb{U})$$

for $0 < |\lambda| \leq 1$.

In view of Lemma 2, we obtain

$$Q(z) \prec 1 + \lambda_1 z \quad (z \in \mathbb{U}) \quad (2.9)$$

and

$$|\lambda_1| = |\lambda| \frac{|\mu|}{|n - \mu|}.$$

Then, we see that

$$\left| \arg \left(f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} \right) \right| < \arctan \left(\frac{|\lambda|}{\sqrt{1 - |\lambda|^2}} \right)$$

and

$$\left| \arg \left(\left(\frac{f(z)}{z} \right)^\mu \right) \right| = \left| \arg \left(\left(\frac{z}{f(z)} \right)^\mu \right) \right| < \arctan \left(\frac{|\lambda_1|}{\sqrt{1 - |\lambda_1|^2}} \right),$$

which give

$$\begin{aligned} \left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| &\leq \left| \arg \left(f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} \right) \right| + \left| \arg \left(\left(\frac{f(z)}{z} \right)^\mu \right) \right| \\ &< \arctan \left(\frac{|\lambda|}{\sqrt{1 - |\lambda|^2}} \right) + \arctan \left(\frac{|\lambda_1|}{\sqrt{1 - |\lambda_1|^2}} \right) \\ &= \arctan \left(\frac{\frac{|\lambda|}{\sqrt{1 - |\lambda|^2}} + \frac{|\lambda_1|}{\sqrt{1 - |\lambda_1|^2}}}{1 - \frac{|\lambda||\lambda_1|}{\sqrt{(1 - |\lambda|^2)(1 - |\lambda_1|^2)}}} \right) \leq \frac{\pi}{2} \quad (z \in \mathbb{U}). \end{aligned}$$

This implies that $f(z) \in \mathcal{S}^*$

On the other hand, when we have

$$1 - \frac{|\lambda||\lambda_1|}{\sqrt{(1 - |\lambda|^2)(1 - |\lambda_1|^2)}} \geq 0,$$

λ satisfies

$$|\lambda| \leq \frac{|n - \mu|}{\sqrt{|n - \mu|^2 + |\mu|^2}} < 1.$$

□

Taking $n = 1$, $0 < \mu < 1$ and $0 < \lambda < 1$, we have the next corollary due to Obradović [4].

Corollary 2. *If $f(z) \in \mathcal{A}$ satisfies the condition*

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} - 1 \right| < \lambda \quad (z \in \mathbb{U})$$

for some $0 < \mu < 1$ and $0 < \lambda < 1$, then $f(z) \in \mathcal{S}^$.*

Applying Lemma 3, we derive the following theorem.

Theorem 2. Let $f(z) \in \mathcal{A}_n$ satisfy the condition

$$f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} \prec 1 + \lambda z \quad (z \in \mathbb{U}) \quad (2.10)$$

for some complex number μ ($\operatorname{Re}(\mu) < \frac{n}{2}$). If the complex number λ_1 is given by

$$|\lambda_1| = |\lambda| \frac{|\mu|}{|n - \mu|},$$

then

$$(1) \quad f(z) \in \mathcal{S}^*(\alpha) \text{ where}$$

$$\alpha \leq \begin{cases} \frac{1 - |\lambda|}{1 + |\lambda_1|} & \left(0 < |\lambda| \leq \frac{|n - \mu|}{|n - \mu| + |\mu|} \right) \\ \frac{1 - (|\lambda|^2 + |\lambda_1|^2)}{2(1 - |\lambda_1|^2)} & \left(\frac{|n - \mu|}{|n - \mu| + |\mu|} \leq |\lambda| \leq \frac{|n - \mu|}{\sqrt{|n - \mu|^2 + |\mu|^2}} \right) \end{cases}$$

and

$$(2) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{(|n - \mu| + |\mu|)|\lambda|}{|n - \mu| - |\mu||\lambda|} \leq 1 \quad (z \in \mathbb{U}),$$

$$\text{where } 0 < |\lambda| \leq \frac{|n - \mu|}{|n - \mu| + 2|\mu|}.$$

Proof. Let us define

$$Q(z) = \left(\frac{z}{f(z)} \right)^\mu,$$

$$p(z) = \frac{zf'(z)}{f(z)},$$

and

$$w(z) = \frac{zf'(z)}{f(z)} - 1.$$

Then by the equation (2.9) of Theorem 1, we have

$$Q(z) \prec 1 + \lambda_1 z \quad (z \in \mathbb{U})$$

where $0 < |\lambda_1| = |\lambda| \frac{|\mu|}{|n - \mu|} < |\lambda| < 1$ since $\operatorname{Re}(\mu) < \frac{n}{2}$. Also, since the condition (2.10) is equivalent to

$$Q(z) \left[\alpha + (1 - \alpha) \frac{p(z) - \alpha}{1 - \alpha} \right] \prec 1 + \lambda z \quad (z \in \mathbb{U})$$

with α is given by (2.4) and as

$$Q(z)[1 + w(z)] \prec 1 + \lambda z \quad (z \in \mathbb{U}),$$

the statement of the theorem directly follows from Lemma 3. \square

Putting $n = 1$, $0 < \mu < \frac{1}{2}$, $0 < \lambda < 1$ and fixed α , we get the following corollary due to Obradović [4].

Corollary 3. Let $f(z) \in \mathcal{A}$ satisfy the condition

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} - 1 \right| < \lambda \quad (z \in \mathbb{U})$$

with some $0 < \mu < \frac{1}{2}$. If the real number λ_1 is given by

$$\lambda_1 = \lambda \frac{\mu}{1-\mu},$$

then

(1) $f(z) \in \mathcal{S}^*(\alpha)$, where

$$\alpha = \begin{cases} \frac{1-\lambda}{1+\lambda_1} & (0 < \lambda \leq 1-\mu) \\ \frac{1-(\lambda^2+\lambda_1^2)}{2(1-\lambda_1^2)} & \left(1-\mu \leq \lambda \leq \frac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}} \right). \end{cases}$$

(2) $\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{\lambda}{1-\mu-\mu\lambda} \leq 1 \quad (z \in \mathbb{U})$,

where $0 < \lambda \leq \frac{1-\mu}{1+\mu}$.

Using Lemma 2 and Lemma 3, we derive

Theorem 3. If $f(z) \in \mathcal{A}_n$ satisfies

$$f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} \prec 1 + \lambda z \quad (z \in \mathbb{U})$$

and

$$F(z) = z \left[\frac{c-\mu}{z^{c-\mu}} \int_0^z \left(\frac{t}{f(t)} \right)^\mu t^{c-\mu-1} dt \right]^{-\frac{1}{\mu}} \quad (z \in \mathbb{U})$$

for some complex numbers λ, μ , and c such that $\operatorname{Re}(c-\mu) < n$, then

(1) $F(z) \in \mathcal{S}^*$ for $|c-\mu||\lambda| \leq \frac{|n-\mu||n-(c-\mu)|}{\sqrt{|n-\mu|^2+|\mu|^2}}$.

(2) $F(z) \in \mathcal{S}^*(\alpha)$ where

$$\alpha \leq \begin{cases} \frac{1-|\lambda_1|}{1+|\lambda_2|} & \left(0 < |\lambda_1| \leq \frac{|n-\mu|}{|n-\mu|+|\mu|} \right) \\ \frac{1-(|\lambda_1|^2+|\lambda_2|^2)}{2(1-|\lambda_2|^2)} & \left(\frac{|n-\mu|}{|n-\mu|+|\mu|} \leq |\lambda_1| \leq \frac{|n-\mu|}{\sqrt{|n-\mu|^2+|\mu|^2}} \right), \end{cases}$$

$$|\lambda_1| = |\lambda| \frac{|c-\mu|}{|n-(c-\mu)|}, \quad |\lambda_2| = |\lambda_1| \frac{|\mu|}{|n-\mu|} \text{ and } \operatorname{Re}(\mu) < \frac{n}{2}.$$

(3) $\left| \frac{zF'(z)}{F(z)} - 1 \right| < \frac{(|n-\mu|+|\mu|)|c-\mu||\lambda|}{|n-(c-\mu)||n-\mu|-|c-\mu||\mu||\lambda|} \leq 1 \quad (z \in \mathbb{U})$

where $|c-\mu||\lambda| \leq \frac{|n-(c-\mu)||n-\mu|}{|n-\mu|+2|\mu|}$ and $\operatorname{Re}(\mu) < \frac{n}{2}$.

Proof. If we put

$$Q(z) = F'(z) \left(\frac{z}{F(z)} \right)^{1+\mu},$$

after some transformations, we obtain

$$Q(z) + \frac{1}{c-\mu} z Q'(z) = f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} \prec 1 + \lambda z.$$

Therefore, spending the same technique as in the proof of Lemma 2, we have that

$$Q(z) \prec 1 + \lambda_2 z \quad (z \in \mathbb{U})$$

and

$$|\lambda_2| = \frac{|\mu| |c - \mu|}{|n - \mu| |n - (c - \mu)|}.$$

The statement of the theorem now easily follows from Theorem 1 and Theorem 2. \square

Taking $n = 1$, $0 < \mu < 1$, $0 < \lambda < 1$ and fixed α , we obtain the next the corollary due to Obradović [4].

Corollary 4. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} - 1 \right| < \lambda \quad (z \in \mathbb{U})$$

and

$$F(z) = z \left[\frac{c-\mu}{z^{c-\mu}} \int_0^z \left(\frac{t}{f(t)} \right)^\mu t^{c-\mu-1} dt \right]^{-\frac{1}{\mu}} \quad (z \in \mathbb{U})$$

for $c - \mu > 0$, then

$$(1) \quad F(z) \in \mathcal{S}^* \text{ for } (c - \mu)\lambda \leq \frac{(1 - \mu)(1 - (c - \mu))}{\sqrt{(1 - \mu)^2 + \mu^2}}.$$

$$(2) \quad F(z) \in \mathcal{S}^*(\alpha) \text{ where}$$

$$\alpha = \begin{cases} \frac{1 - \lambda_1}{1 + \lambda_2} & (0 < \lambda_1 \leq 1 - \mu) \\ \frac{1 - (\lambda_1^2 + \lambda_2^2)}{2(1 - \lambda_2^2)} & \left(1 - \mu \leq \lambda_1 \leq \frac{1 - \mu}{\sqrt{(1 - \mu)^2 + \mu^2}} \right), \end{cases}$$

$$\lambda_1 = \lambda \frac{c - \mu}{1 - (c - \mu)}, \lambda_2 = \lambda_1 \frac{\mu}{1 - \mu} \text{ and } 0 < \mu < \frac{1}{2}.$$

$$(3) \quad \left| \frac{z F'(z)}{F(z)} - 1 \right| < \frac{(c - \mu)\lambda}{(1 - (c - \mu))(1 - \mu) - (c - \mu)\mu\lambda} \leq 1 \quad (z \in \mathbb{U}) \text{ where}$$

$$0 < \lambda \leq \frac{(1 - (c - \mu))(1 - \mu)}{(c - \mu)(1 + \mu)} \text{ and } 0 < \mu < \frac{1}{2}.$$

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